Course of

Active Control of

Noise and Vibrations

An Improved Lattice-Based Adaptive IIR Notch Filter

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Introduction

The main purpose of this project was to produce a digital notch filter that is able to adapt its coefficients in order to obtain a specified behavior. Adaptive notch filtering is a well-studied technique for removing or retrieving sinusoids of unknown frequencies from additive broadband noise. In general a notch transfer function has two or more coefficients determining bandwidth and peak frequency of the notch. To obtain coefficients adaptation lot of methods can be used, between which we propose: an alternate network structure (par. 2), a structurally induced bandpass realization (par. 3) and a planar-rotation lattice filter based structure, that is the main topic of this paper. Instead of minimizing an output error cost function or following a gradient descendent procedure, our filter is designed to achieve a stable associated differential equation, producing advantages as unbiased frequency estimation and faster convergence.

This report is organized as follows.

Section 1 introduces to the argument of notch filters.

Section 2 illustrates the lattice structure proposed by N. I. Cho, C. H. Choi and A. U. Lee and is almost entirely taken from [1].

Section 3 describes the advantages of using biquads (bilinearly transformed second order) filters and is almost entirely taken from [2].

Section 4 proposes the main topic of this paper, describing the notch filter, invented by P. A. Regalia, that makes use of a second order lattice filter obtained from a particular planar rotation structure. Also this chapter is almost entirely taken from the bibliography [4].

Section 5 makes some comparative simulation between the previous filters, including a brief description of advantages of the proposed one.

Section 6 includes other results of simulations made with the Regalia's filter, in particular exploring the possibilities of forming series or parallel connections of this filter in case of multiple frequency input signals, and then makes some conclusions.

1. About Notch Filters

The ideal notch transfer function has the following frequency response:

$$F_{\text{ideal}}(e^{j\omega}) = \begin{cases} 0, & \omega = \omega_k \\ 1, & \text{otherwise} \end{cases}$$

A standard approximation is this one:

$$F(z) = \frac{N(z)}{N(z/\rho)}, \qquad 0 \ll \rho < 1$$

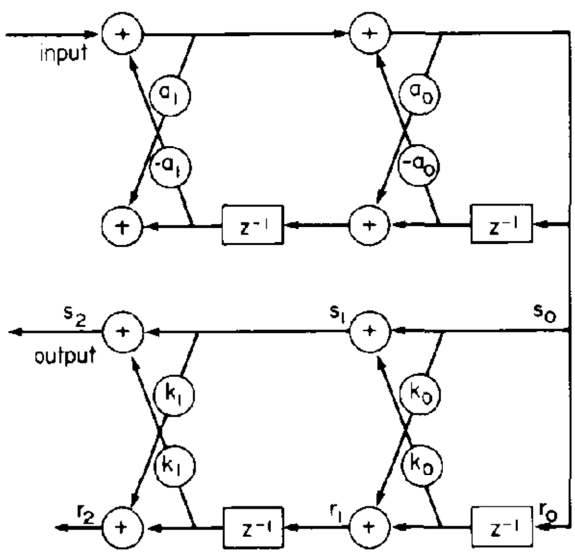
Where *N(z)* is a polynomial with its zeros on the unit circle and thus *1/N(z)* is a all-pole filter with poles near the unit circle. This kind of structure manifest two great weaknesses:

1) The poles near the unit circle make it possible that, using finite precision instruments like computers, a data approximation may introduce a little shift outside the stability circle, making the system explode.
2) If the gain ρ is chosen in order to maintain the poles inside the unit circle, this can introduce an undesired bias in the frequency estimation.

Two solutions of this problem are presented in the next two chapters, both making use of a IIR structure preserving stability constraints.

2. IIR lattice notch filter based on the cascade of two allpass filters

In 1988 N. I. Cho, C. H. Choi and A. U. Lee proposed a new, very simple ALE (Adaptive Line Enhancer) employing a lattice-type notch filter, which is adapted using adaptive algorithms related to the lattice FIR filters. Two algorithms were proposed: the first algorithm adapts *2p* coefficients for *p* sinusoids, the second one adapts only one coefficient per sinusoid restricting the zeros of the filters to the unit circle.
They employed a lattice IIR filter which can be viewed as a cascade of all-pole and all-zero lattice filters, as shown in the figure below.



Nam Cho lattice structure

The transfer function of the filter is

$$H(z) = \frac{1 + k_0(1 + k_1)z^{-1} + k_1 z^{-2}}{1 + a_0(1 + a_1)z^{-1} + a_1 z^{-2}}$$

where a_0 and a_1 can be expressed in terms of k_0 and k_1, obtaining only 2 coefficients as mentioned before. In particular if α is close to 1, the following approximation hold:

$$a_1=\alpha k_1$$
$$a_0=k_0$$

Stability conditions imposed $|a_0|<1$ and $|a_1|<1$. This is true if $|k_0|$ and $|k_1|$ are less than 1. As standard algorithms adapt filter coefficients so as to minimize the mean-squared values of the signals in the next stage, they adapted k_0 and k_1 to minimize respectively the squared value of s_1, r_1 and s_2, r_2. The following recursive equations are then derived:

$k_m(n) = -\frac{C_m(n)}{D_m(n)}$

$C_m(n) = \lambda C_m(n-1) + s_m(n) r_m(n-1)$

$D_m(n) = \lambda D_m(n-1) + 0.5 s_m^2(n) + 0.5 r_m^2(n-1) \qquad (m = 0{,}1 \; 0 < \lambda < 1)$

where λ is a forgetting factor.
Restricting the zeros of the lattice filter of Fig. to the unit circle implies fixing k_1 to 1. Now k_0 becomes the only parameter to be adapted, in order to minimize the time average of $s_2^2(n)$ and $r_2^2(n)$. As can be shown, if k_1=1 then r_2=s_2, implying that only one of them has to be taken into account. We can express s_2 in terms of s_0 and r_0 as follows:

$$s_2(n) = k_0(n-1)s_0(n-1) + k_0(n)r_0(n-1) + r_0(n-2) + s_0(n).$$

It can be shown that the value of k_0 that minimizes the time average of $s_2^2(n)$ can be computed recursively by

$k_0(n) = -\frac{C(n)}{D(n)}$

$C(n) = \lambda C(n-1) + A(n)B(n)$

$D(n) = \lambda D(n-1) + A^2(n) \qquad (0 < \lambda < 1)$

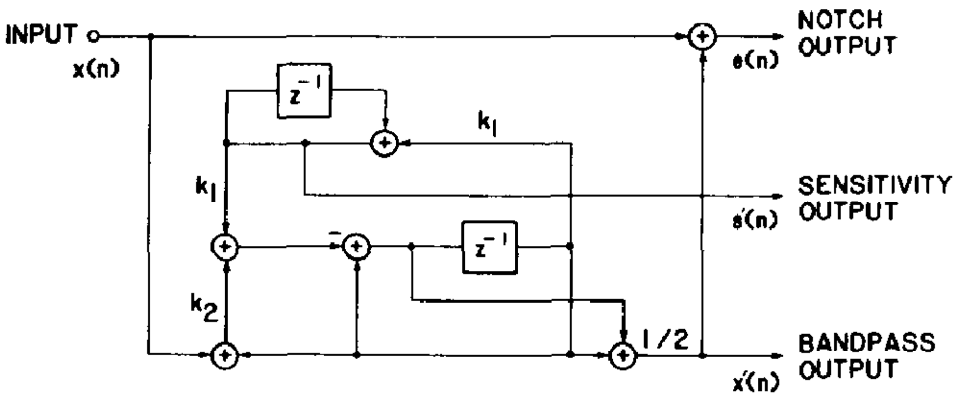
Because in this case is not guaranteed that $|D(n)|$ will always be greater than $|C(n)|$, it is possible that k_0 becomes greater than 1. To maintain the stability constraints is thus necessary to use a smoothed version of k_0, that we call k_0' and is obtained as

$$k_0'(n) = (1-\gamma)k_0'(n-1) + \gamma k_0(n) \qquad (\gamma\text{=0.5})$$

In conclusion this method requires four additional states in addition to four internal states for the all-pole/all-zero cascade, while the method we are going to propose needs only two internal states as will be shown in Ch.

3. Detection of sinusoids using a cascade of bilinear notch filters

This kind of approach was proposed by T. Kwan and K. Martin in 1989. It makes use of biquads (bilinearly transformed second order filters): first a bilinear bandpass filter is realized, then a bilinear notch filter is obtained subtracting the input and the output of the bandpass filter. This well-known technique was usually adapted following the Gauss-Newton Update, that is a gradient based method that requires a computationally expensive matrix inversion. The algorithm they proposed had the purpose of obtaining almost the same behavior of Gauss-Newton one without matrix inversions.
The proposed notch filter has the structure shown below:



Kwan constant bandwidth biquad

The transfer function from the input to the bandpass output is given by

$$H_{bp}(z) = \frac{-k_2}{2} \frac{(1+z^{-1})(1-z^{-1})}{1-(2-k_2-k_1^2)z^{-1}+(1-k_2)z^{-2}}$$

Where k_1 and k_2 are the filter parameters that control respectively the peak frequency and the bandwidth of the filter. These two parameters also make it possible to easily and precisely determine the frequency of the sinusoids without any computational expensive Fourier Transform, as given by the formula

$$\omega_{peak} = 2\arcsin\left(\frac{k_1}{2\sqrt{1-\frac{k_2}{2}}}\right).$$

For k_1 and k_2 small

$$\omega_{peak} = k_1 \left(1 + \frac{k_2}{4}\right).$$

By keeping k_2 fixed, they were able to track different resonant frequencies maintaining the bandwidth of the filter constant. The proposed update law needs an additional filter in order to obtain a sensitivity function that conveys informations about the first order derivative (or gradient) of tho output error power with respect to changes in the filter coefficients. This new filter can be obtained from the previous one by taking the first filter bandpass output as input and the intermediate stage named $s'(n)$ in Fig. as sensitivity output. The resulting update equation is the one shown below:

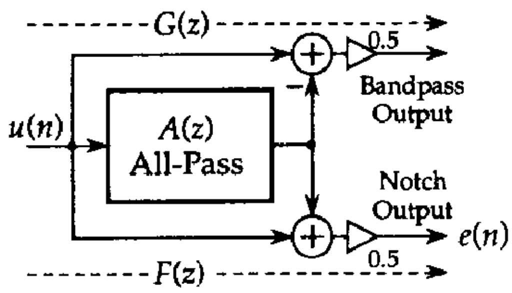
$$k_1(n+1) = k_1(n) - \mu \frac{e(n)s'(n)}{||s'(n)||^2 + P_{min}}$$

Where $e(n)$ is the first filter notch output , $||s'(n)||^2$ can be approximated by squaring the instantaneous value $s'(n)$ and passing the result through a lowpass filter and P_{min} is a constant that prevent division by zero (typically can be taken between 0.001 and 0.01).
We can conclude that this process is, for small changes around the optimum solution, a good approximation of Newton-Gauss gradient based method, but we must also take into account that the update portion needs a second filter of the same complexity than the notch one and that for extremal frequencies this algorithm manifests great difficulties, as shown in Ch.

4. The proposed lattice structure

In 1991 A. Regalia proposed then a new technique for retrieving sinusoids that, starting from the knowledge that lays at the base of the previous examples, is greatly more efficient and less expensive.
As mentioned in the Introductions, this algorithm is designed to achieve a stable differential equation associated to the update law instead of minimize an output error cost function. The realization is simpler than the lattice scheme proposed in Ch.2, as no FIR postfilter is necessary, and simpler than the adaptive bandpass realization of Ch.3, as no additional gradient filters are necessary.
The core of this structure is explained in the rows that follows.

As already seen in previous examples, a well-known technique for obtaining simultaneously a notch and a bandpass filter is to pass the input signal into a allpass structure and then sum or subtract the output of this structure in respect to the original input signal, as shown in next figure.

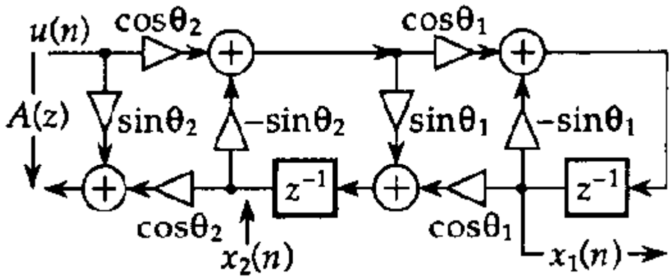


Regalia notch filter scheme

Calling *A(z)* the allpass transfer function, we obtain the notch and bandpass outputs respectively as:

$$F(z) = \frac{1}{2}[1 + A(z)] \quad \text{and} \quad G(z) = \frac{1}{2}[1 - A(z)]$$

We choose *A(z)* as the second order lattice filter obtained from the planar rotation structure shown in the figure that follows.



The allpass filter structure

Here each rotation angle θ_k (k=1, 2) become the adjustable parameters in an adaptive realization. θ_1 controls the notch frequency ω_0 and θ_2 controls the notch bandwidth *B*, as can be seen in these two equations:

$$\theta_1 = \omega_0 - \frac{\pi}{2}, \qquad \omega_0 \in [0, \pi]$$

$$\sin\theta_2 = \frac{1 - \tan\left(\frac{B}{2}\right)}{1 + \tan\left(\frac{B}{2}\right)}$$

To tune the notch frequency parameter θ_1 the following algorithm is proposed:

$$\theta_1(n+1) = \theta_1(n) - \mu(n)e(n)x_1(n)$$

where *e(n)* is the output of the filter F(z) and $x_1(n)$ and $\mu(n)$ can be obtained by

$$x_1(n) = \frac{z^{-1}\cos\theta_2\cos\theta_1}{1 + \sin\theta_1(1 + \sin\theta_2)z^{-1} + \sin\theta_2 z^{-2}} u(n)$$

$$\mu(n) = \frac{1}{\sum_{k=0}^{n} \lambda^{n-k} x_1^2(n)}$$

being λ a forgetting factor.

Notice that the product $e(n)x_1(n)$ is not an estimate of the gradient $\frac{\partial E[e^2(n)]}{\partial \theta_1}$, and is not a minimization of an output error cost function. Instead, this update equation is chosen in order to obtain a stable associated differential equation

$$\frac{d\theta_1(\tau)}{d\tau} = -E[e(n)x_1(n)]$$

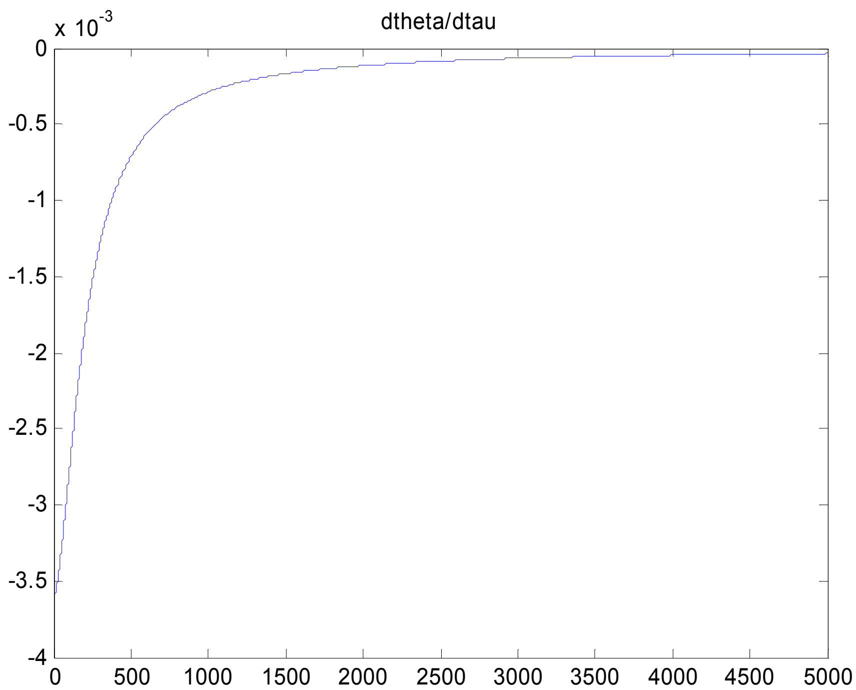
In particular, some calculations will show that

$$E[e(n)x_1(n)] = -\frac{\alpha_1^2}{4\pi} \frac{cos\theta_2(\tau)cos\theta_1(\tau)}{|1 + sin\theta_1(\tau-1)(1+sin\theta_2(\tau-1))+sin\theta_2(\tau-2)|^2} [cos\omega_1 + \sin\theta_1(\tau)].$$

Where α_1 and ω_1 are the amplitude and angular frequency of the input sinusoid.
This equation is asymptotically stable to a stationary position satisfying

$$\sin\theta_1(\infty) = -cos\omega_1.$$

As can be seen in next figure, the value of the associated equation tends to zero after few samples, indicating that convergence is always reachable.



Convergence analysis

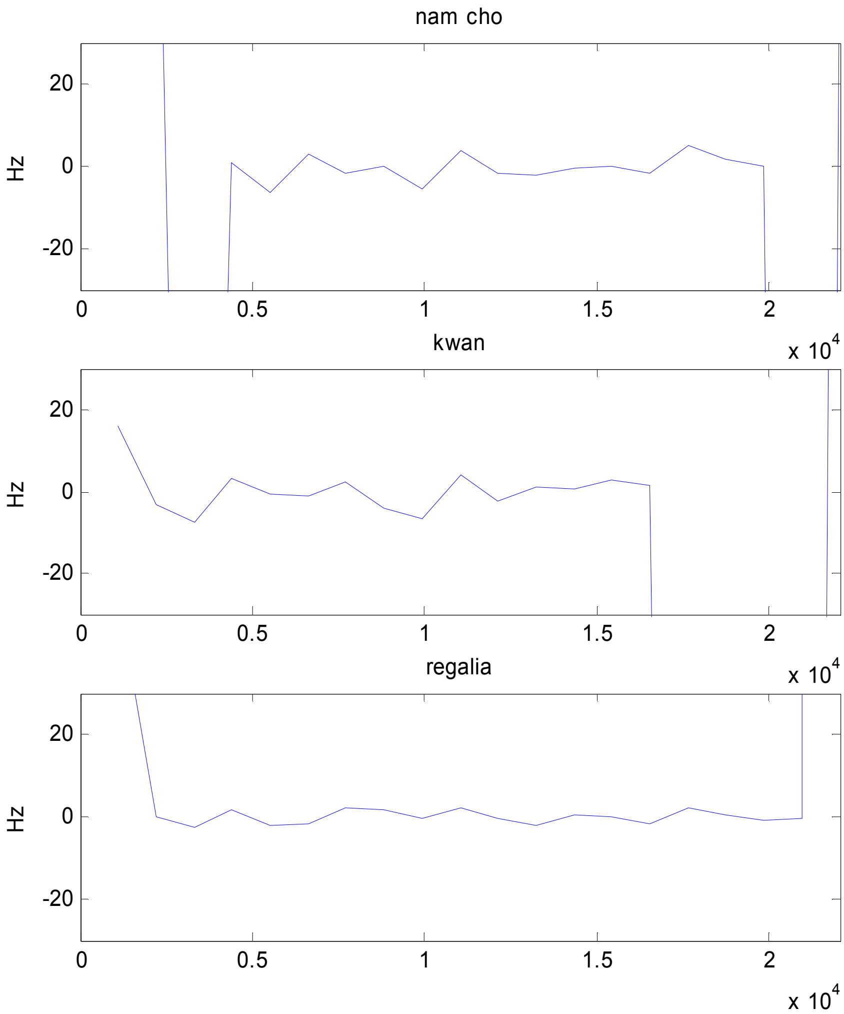
In conclusion we propose a method that requires only two internal states and no additional postfilters. In addition to this, the implemented system shows a very better behavior at critical frequencies than filters presented before, as we can see in next Chapter.

5. Competitive simulation examples

Simulation results are presented as comparison between the three algorithms seen before. For consistency, the three schemes are adapted using the same adaptation parameters:

- the stepsize $\mu(n)$ is adapted following the iterative adaptation seen in Ch. 4 (replacing $x_1(n)$ by $s'(n)$ where necessary);
- the pole radius is set for all the algorithms to a value of 0.97;
- the forgetting factor λ is chosen equal to 0.99.

Next figure shows the frequency estimation error (in Hertz) after convergence.



Frequency estimation after convergence

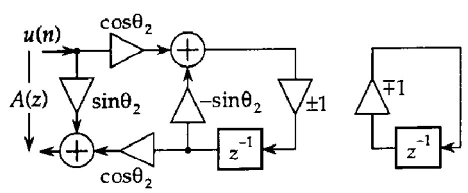
Comparison results are taken for 20 equidistant frequencies, from 0 to Nyquist frequency. For each frequency *i* the input signal $s_i(n)$ was corrupted by the same additive white noise s_1, as can be seen below:

```
s(i,:)=alpha(i)*sin(omega(i)*t + phi(i))+s1;
```

where α is the frequency amplitude, randomly chosen in {1, 2, …., 10}, ω_i is the i-th frequency and ϕ_i is the i-th phase shift. The estimates were time averaged from 100 samples, after convergence is reached. As we can see, all three schemes yield acceptable accuracy for midband frequencies, but only the proposed algorithm achieve good estimation also at near critical frequencies. Three main advantages of our method are shown:

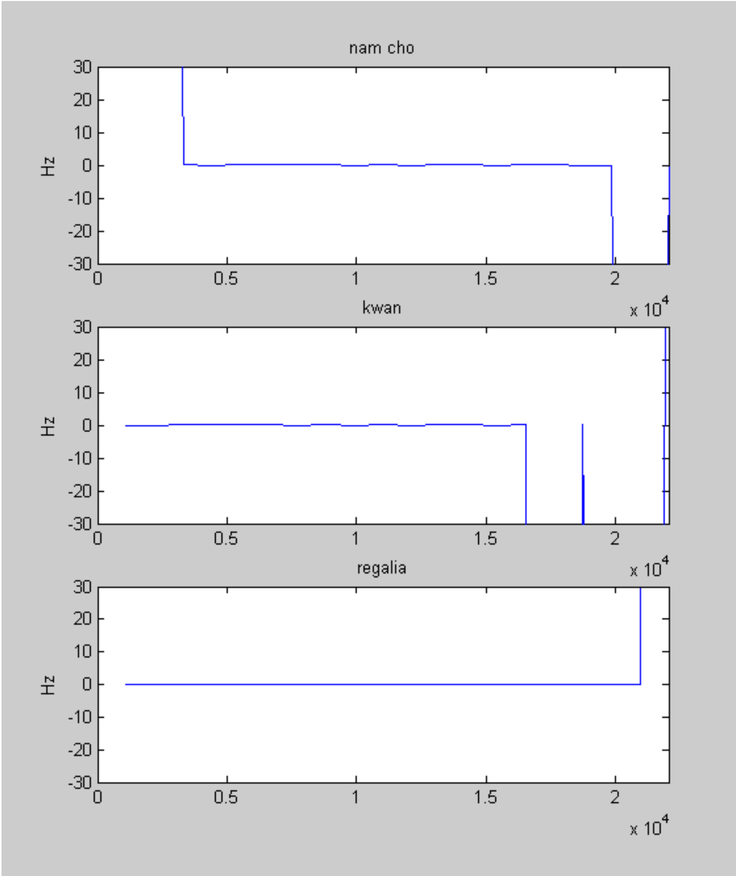
1) while the scheme proposed by Kwan requires a large initial bandwidth, our approach is able to track sinusoids very distant from the original notch position even if the notch bandwidth is short. This fact is due to the cascade of bandpass and all-pole filters in the first one, that make the input sinusoid give a comparatively small contribution to the gradient filter output. Instead, the Regalia scheme, achieve good convergence because the regressor signal $x_1(n)$ is obtained from an all-pole (only) filtered version of the input, making this last a bigger contribution to the coefficient adaptation.

3) In limiting case, when the frequency of the input becomes equal to the Nyquist limit, the allpass portion of the Regalia scheme degenerates to the one shown in next figure, and the pole at z=1 is both unobservable and uncontrollable. In Kwan method, instead, pole z=1 is still unobservable, but remains controllable and thus is susceptible of unbounded growth from input excitation, making the system in fact unstable.



Degenerated Regalia filter

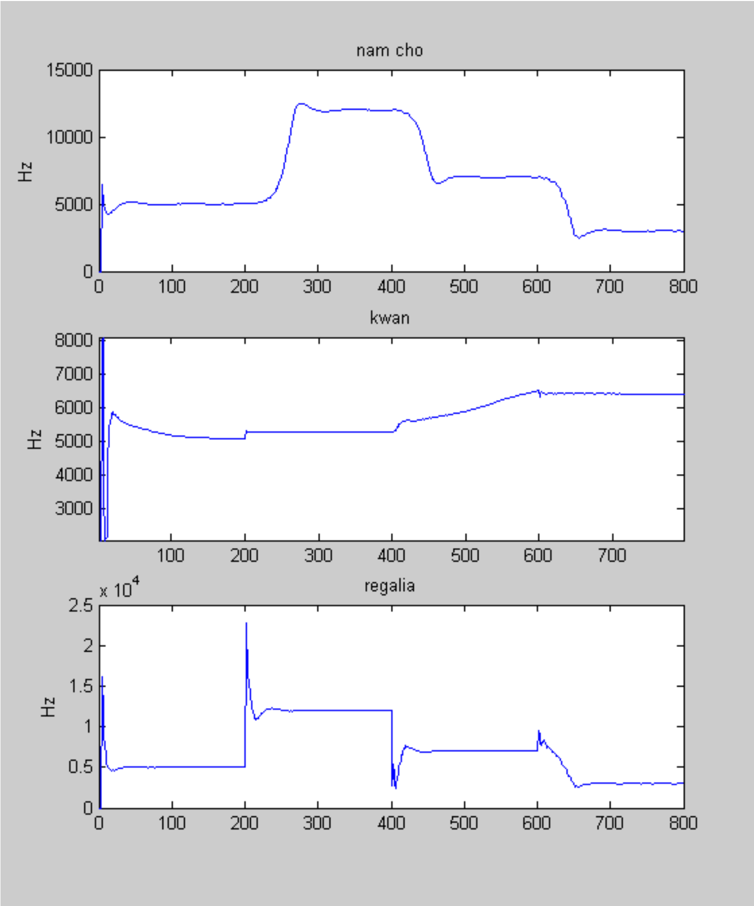
This behaviour near critical frequencies can be seen in the following picture, where only the sinusoidal components are present, while the white noise component has been removed.



Extreme frequencies behaviour (in absence of noise)

4) The proposed scheme simply requires a difference operation to obtain the instant frequency estimation, while the other two require inverse trigonometric relations.

A second simulation is proposed, where the input frequency is changed every 200 samples. The frequency signal was corrupted by additive white noise, with SNR of 3dB. The forgetting factor λ is fixed to 0.97, the pole radius is chosen equal to 0.92. As can be seen in next figure, the proposed scheme shows superior tracking of the one proposed by Kwan, because its regressor signal $x_1(n)$ is obtained from an all-pole (rather than an allpass-allpole cascade) filtered version of the input signal. Thus the sinusoidal component exerts a sufficient influence on the parameter evolution when the notch frequency is displaced from the signal frequency (as already seen in Par.1 of this chapter).



Speed of convergence in case of changing frequency input

6. Other Results

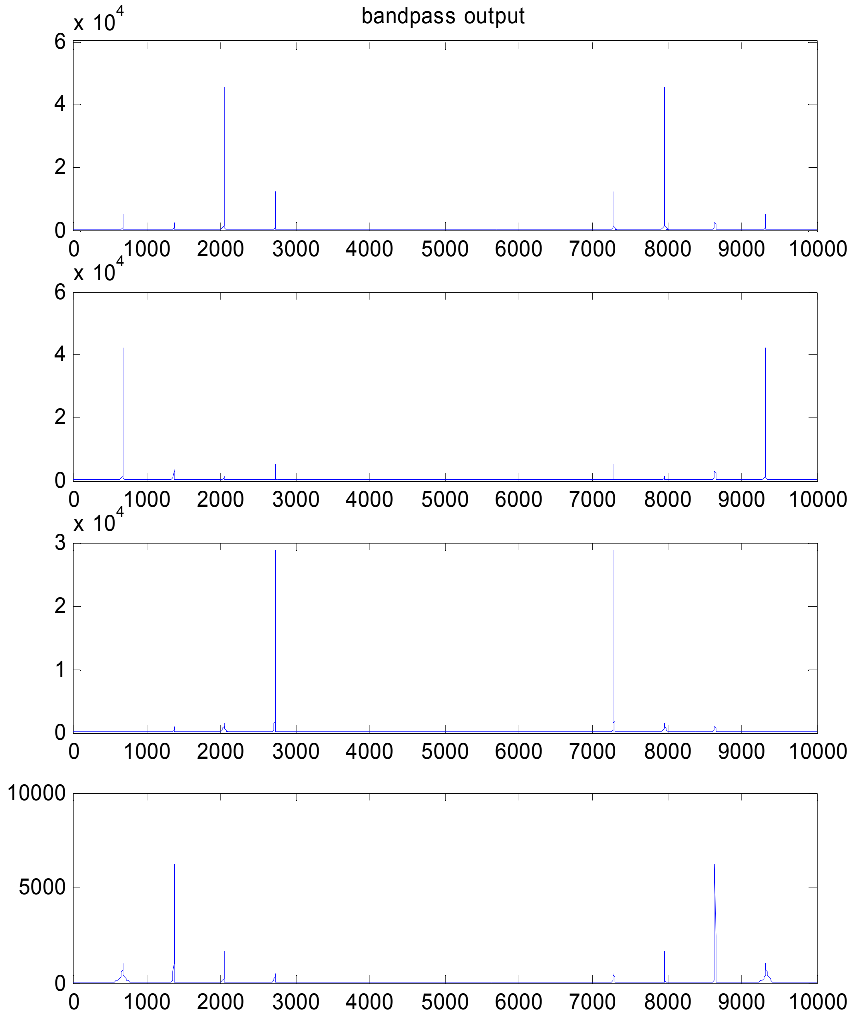
As final results, we propose three practical implementations using combinations of N of the filters inspired by Regalia. For each of them a simulation of extraction of multiple sinusoids is presented. The algorithms take as input signal a signal obtained from the summation of N equally spaced sinusoids of equation

```
s(i,:)=alpha(i)*sin(omega(i)*t + phi(i));
```

adding then to them a white noise as disturb.

Subsequent application of the proposed scheme

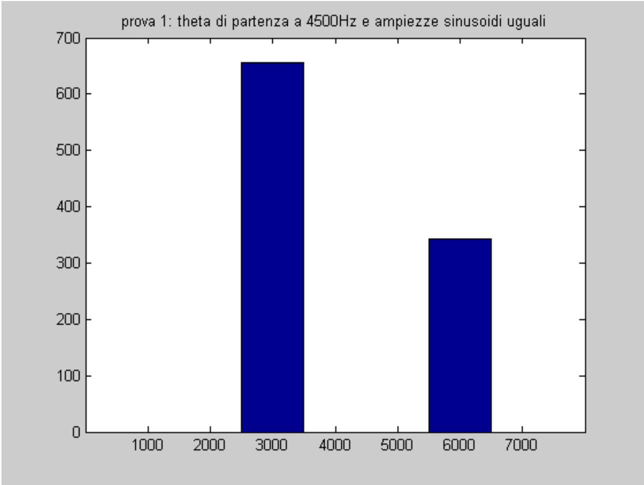
The first, proposed process of extraction of the N sinusoids is generated as a successive repetition of the single sinusoid method, letting the notch output become the new filter input, once that convergence is reached. The result is that each filter extract a single sinusoid, letting the other pass (if the notch bandwidth is strict). At the output of the last notch only the white noise is present.



Performance analysis in case of only one filter for multiple sinusoids

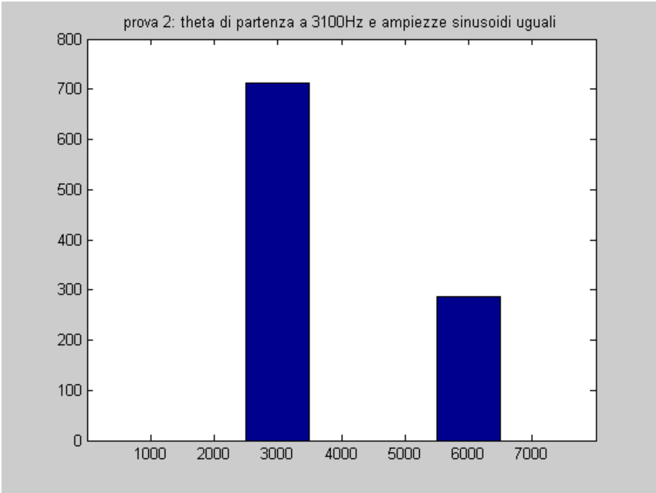
As second point of this last chapter, we propose a statistical approach in analyzing how the single-filter structure behaves in presence of multiple sinusoids as input signal *u(n)*. As results of this step, we found six possibilities that can be investigated. Starting from a signal *u(n)* formed by white noise added to only two sinusoids of frequency 3000Hz and 6000Hz, we have proposed six experimentations and results report the number of times the system converge toward one frequency over 1000 simulations:

1) the starting θ_1 is taken at 4500Hz and sinusoids amplitude is the same;



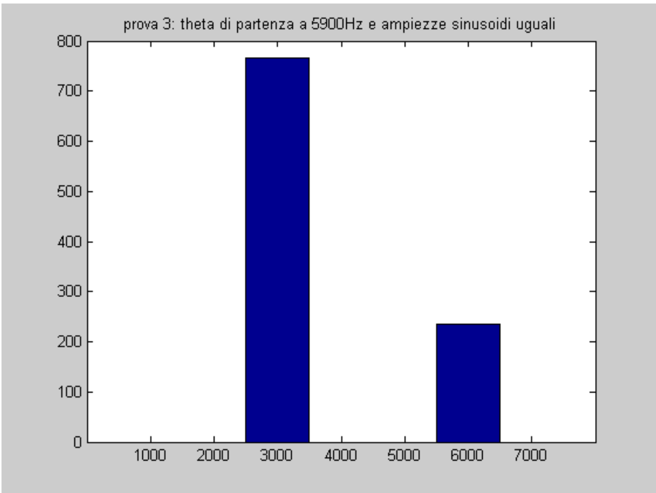
Statistical results over 1000 sample tests

2) the starting θ_1 is taken at 3100Hz and sinusoids amplitude is the same;



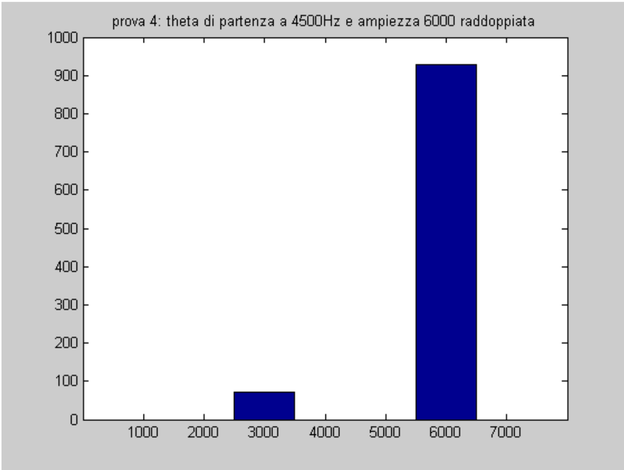
Statistical results over 1000 sample tests

3) the starting θ_1 is taken at 5900Hz and sinusoids amplitude is the same;



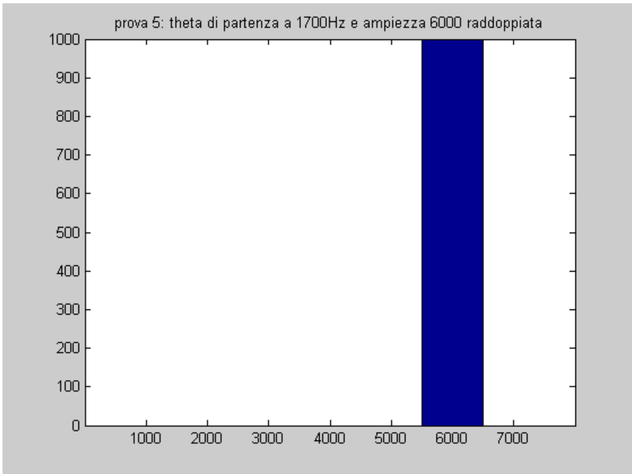
Statistical results over 1000 sample tests

4) the starting θ_1 is taken at 4500Hz and the amplitude of 6000Hz sinusoid is doubled;



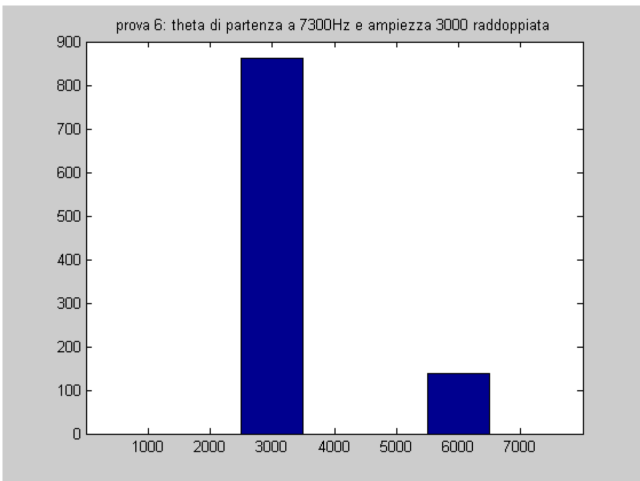
Statistical results over 1000 sample tests

5) the starting θ_1 is taken at 1700Hz and the amplitude of 6000Hz sinusoid is doubled;



Statistical results over 1000 sample tests

6) the starting θ_1 is taken at 7300Hz and the amplitude of 3000Hz sinusoid is doubled.

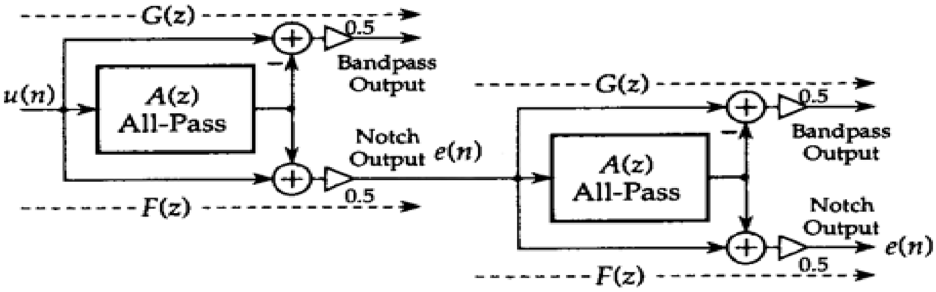


Statistical results over 1000 sample tests

We can simply read from this figures that the presence of two equal-amplitude sinusoids converges towards one of them without a specific rule; if instead one of the two sinusoids has amplitude that is the double of the other's, the system converges towards that sinusoid in almost all cases.

Series of two filters

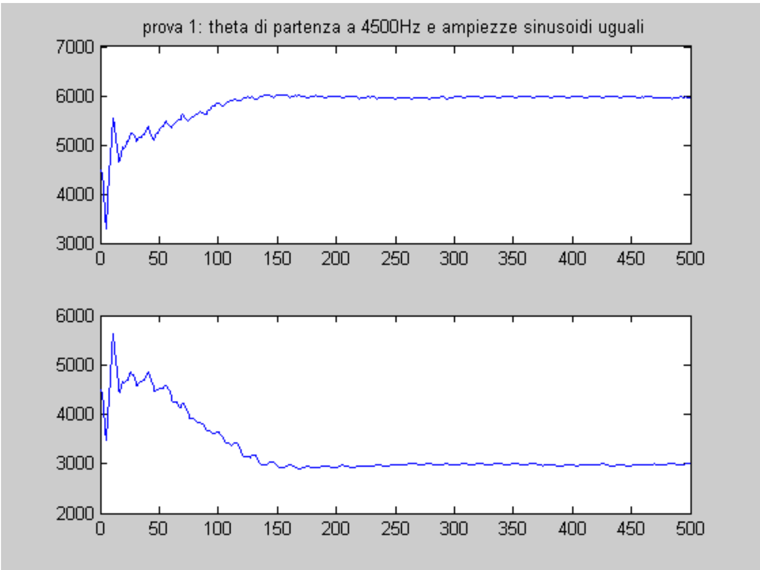
The tested scheme was this following one:



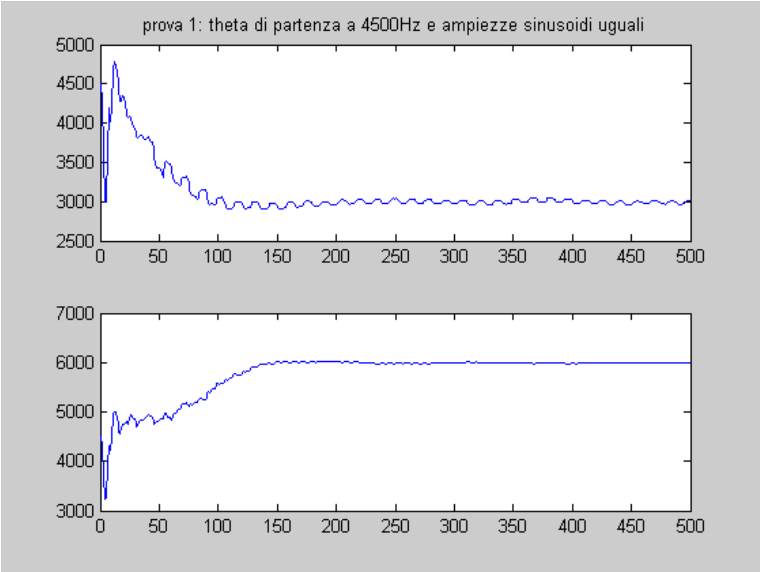
Two-filter series

The input signal *u(n)* is again obtained from the sum of a white noise of amplitude m_u and the two sinusoids seen before. Three cases of study are proposed and for each of them simulations with m_u =1 and m_u =0.1 are done:

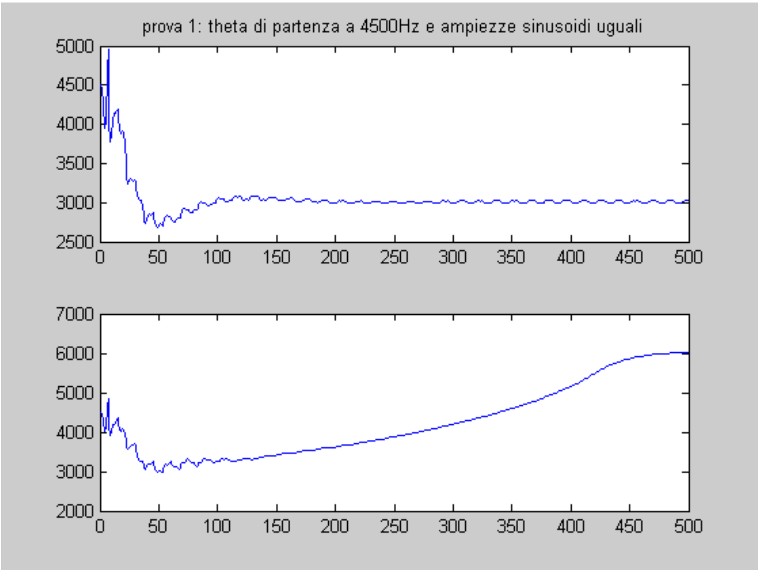
1) the starting θ_1 of both filters is taken at 4500Hz and sinusoids amplitude is the same;



Frequency estimation of the two filters for m_u =1

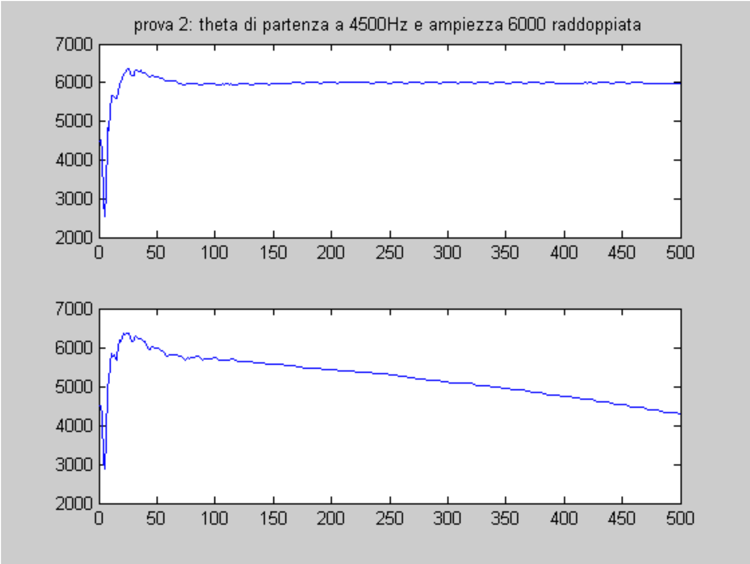


Frequency estimation of the two filters for m_u =1 (dual case)

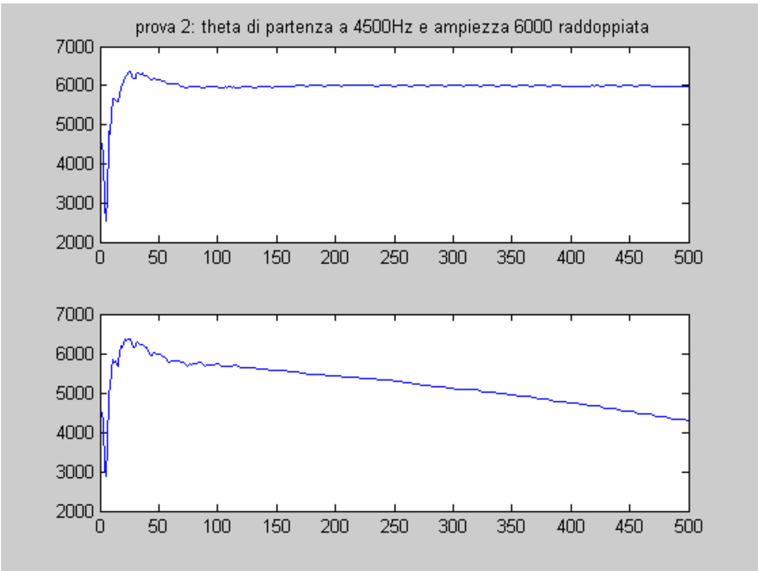


Frequency estimation of the two filters for m_u =0.1

2) the starting θ_1 of both filters is taken at 4500Hz and the amplitude of 6000Hz sinusoid is doubled;

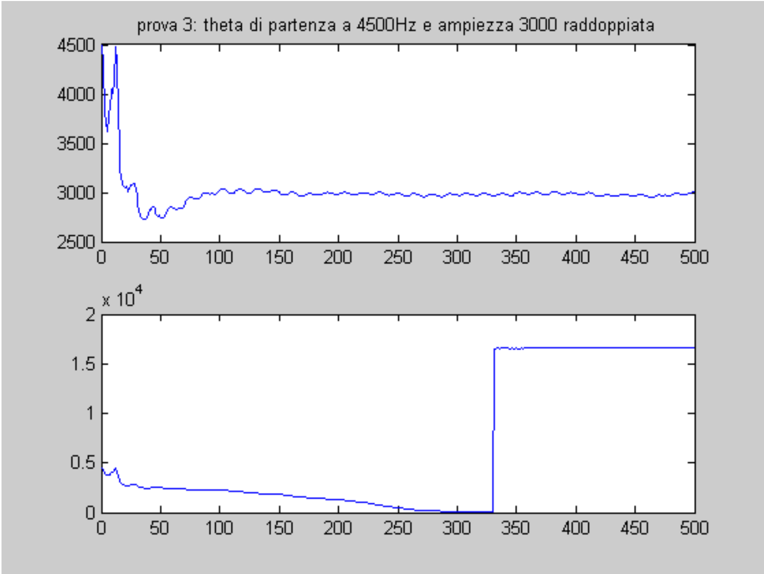


Frequency estimation of the two filters for m_u *=1*

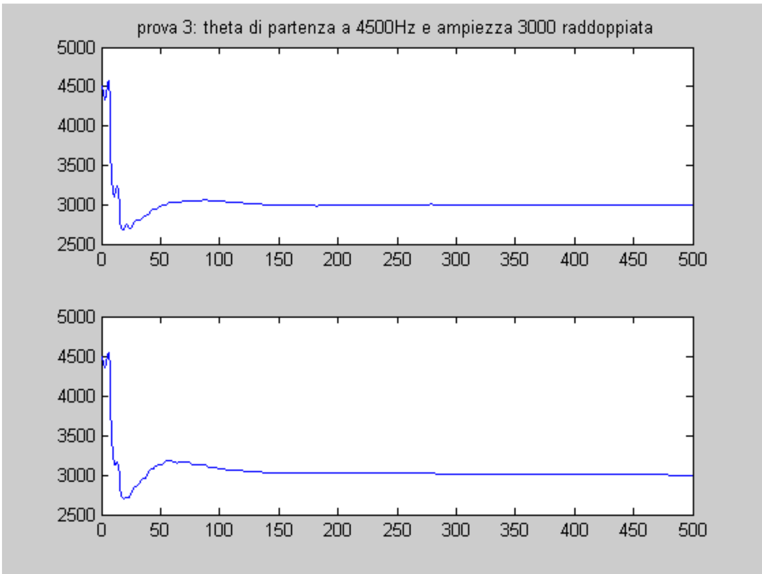


Frequency estimation of the two filters for m_u *=0.1*

3) the starting θ_1 of both filters is taken at 4500Hz and the amplitude of 3000Hz sinusoid is doubled;



Frequency estimation of the two filters for m_u =1

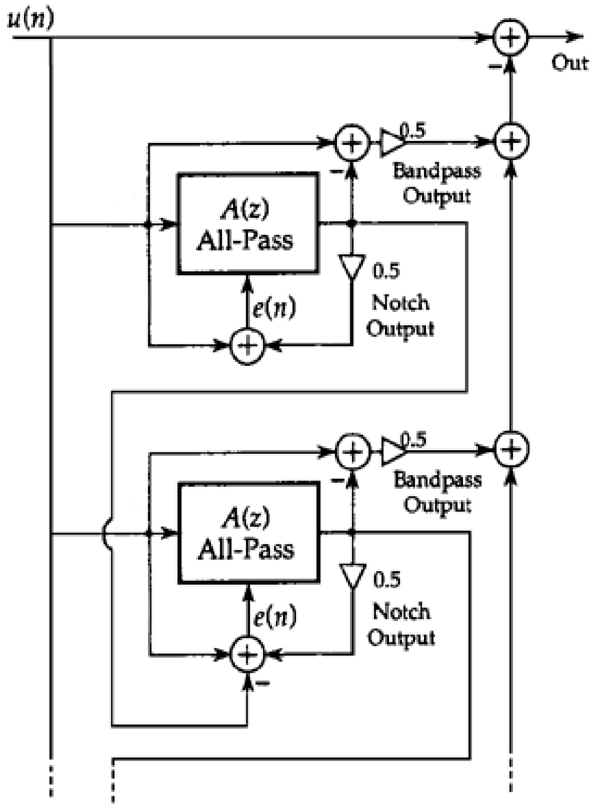


Frequency estimation of the two filters for m_u =0.1

It is visible that in the first case the two filters converge one towards one frequency and the other towards the second one, presenting the m_u =0.1 test more difficulties, because of more fast convergence towards the same frequency and then a slow-time correction. In the second and the third experimentations, the second filter convergence is deeply influenced by the first filter presence: in the second case a very slow coefficient adaptation due to step-size adaptation seen in Ch. 4 results, while a sort of instability is shown in the third and last case, in which only a decrement of m_u can prevent this sort of behavior.

Parallel of two filters

The tested scheme was this following one:

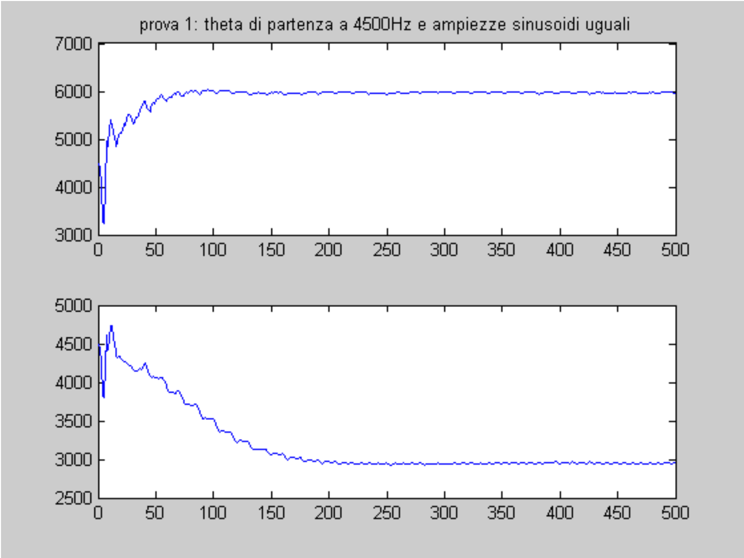


Two-filter parallel scheme

where N Regalia filters are collected in parallel and the e(n) of each filter (except the first one) is the result of the subtraction of the usual and the previous filters bandpass outputs.

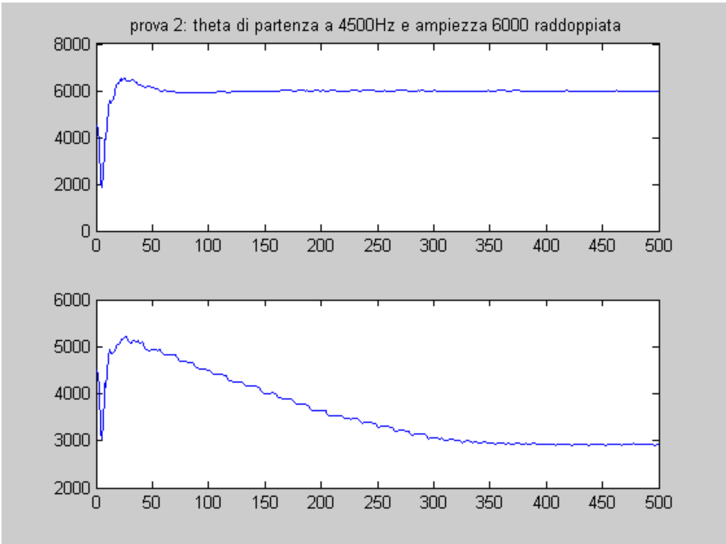
The input signal *u(n)* is one more time obtained from the sum of a white noise and two sinusoids of 3000Hz and 6000Hz. The same three cases seen before are studied. The results are the following:

1) the starting θ_1 of both filters is taken at 4500Hz and sinusoids amplitude is the same;



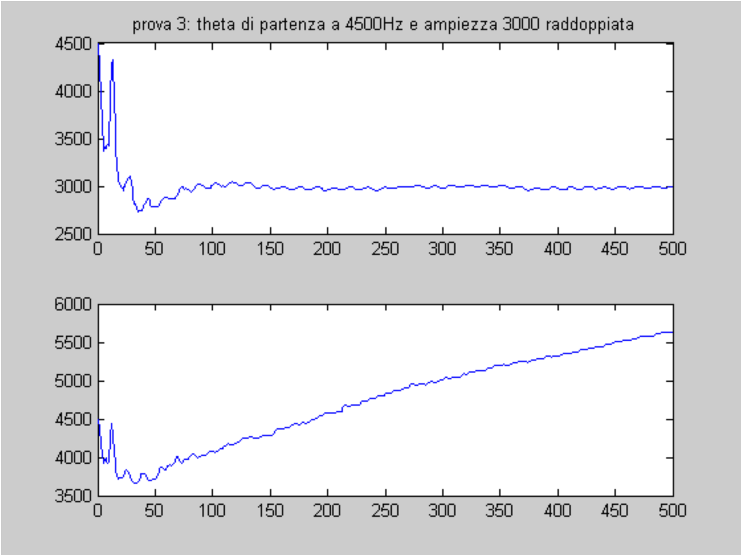
Frequency estimation of the two filters

2) the starting θ_1 of both filters is taken at 4500Hz and the amplitude of 6000Hz sinusoid is doubled;



Frequency estimation of the two filters

3) the starting θ_1 of both filters is taken at 4500Hz and the amplitude of 3000Hz sinusoid is doubled.



Frequency estimation of the two filters

As can be evident from this figures, both frequencies are extracted because each adaptation signal *e(n)* does contain all information about the current filter without the presence of the frequency towards which previous filters are converging. This results are the best part of this project, making us now able to use this structure to obtain a simultaneous multiple frequency band pass filter or, as shown in the scheme seen before, a simultaneous multiple frequency notch filter.

References

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